GOVERNMENT DEGREE COLLEGE, RAJAMPETA

DEPARTMENT OF MATHEMATICS

ADDITIONAL INPUTS

Ring Theory

Unit 1: Rings, Fields, and Integral Domains

Lecture Input: Beyond definitions, focus on the *Characteristic* of a ring and how it dictates the ring's behavior. IIT JAM often asks about finite fields (Galois Fields) implicitly.

Q1: Concept of Characteristic

Question: Let R be a ring with unity of characteristic 3. For any $a, b \in R$, what is the value of $(a+b)^3$?

(A) $a^3 + b^3$

(B) $a^3 + 3a^2b + 3ab^2 + b^3$

(C) $a^3 + b^3$ only if R is commutative

(D) $a^3 + b^3$ regardless of commutativity

Solution:

- Step 1: By the binomial theorem, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
- Step 2: Characteristic 3 means $3x = 0$ for all $x \in R$ (i.e., $3a^2b = 0$ and $3ab^2 = 0$).
- Step 3: The expression simplifies to $a^3 + b^3$. Note: We did not swap a and b to cancel terms; the middle terms vanished due to the characteristic. This holds in *any* ring of char 3.
- Answer: (A) $a^3 + b^3$

Q2: Zero Divisors in Matrix Rings (IIT JAM 2019 Trend)

Question: Consider the ring $M_2(\mathbb{R})$ of 2x2 real matrices. Which of the following is a zero divisor?

(A) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ (C) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ (D) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Solution:

- Step 1: A zero divisor A satisfies $A \neq 0$ and exists $B \neq 0$ such that $AB = 0$.
- Step 2: In matrices, zero divisors are precisely *singular* matrices (determinant = 0).

- Step 3: Check determinants: (A) det=2, (B) det=0, (C) det=1, (D) det=4.
- Answer: (B)

Unit 2: Subrings and Ideals

Lecture Input: The key distinction is that *Ideals* are "super-subrings" (closed under multiplication by the *whole* ring, not just the subset). This is the hardest jump for B.Sc. students to visualize.

Q3: Necessity of "1" in Subrings

Question: Let $R = \mathbb{Z}_6$ (integers modulo 6). Consider $S = \{0,2,4\}$. Is S a subring? Is it an ideal?

(A) Subring only (B) Ideal only (C) Both (D) Neither

Solution:

- Step 1: Check Subring: $2+4=0$, $2+2=4$. Closed under addition. $2 \times 4 = 8 \equiv 2$. Closed under multiplication. Contains 0. It is a subring (Note: It does *not* need to contain '1' to be a subring, though your book defines it with 1; many standard exams define subring without requiring 1).
- Step 2: Check Ideal: Take $2 \in S$ and $1 \in R$. $2 \times 1 = 2 \in S$. Take $2 \in S$ and $3 \in R$. $2 \times 3 = 6 \equiv 0 \in S$.
- Step 3: Wait—Take $4 \in S$ and $3 \in R$. $4 \times 3 = 12 \equiv 0 \in S$. It passes. Actually, S is the ideal generated by 2 (i.e., $2\mathbb{Z}_6$).
- Answer: (C) Both

Q4: The "Center" is a Subring

Question: The *center* of a ring R is defined as $Z(R) = \{x \in R : xr = rx \quad \forall r \in R\}$. If R is a division ring (like Quaternions), $Z(R)$ is:

(A) An ideal (B) A subfield (C) A zero divisor (D) Not a subring

Solution:

- Step 1: For Quaternions $\mathbb{H}$, the center is $\mathbb{R}$ (the real numbers).
- Step 2: $\mathbb{R}$ is a field. Since R is a division ring, its center is always a *field* (commutative division ring).
- Answer: (B) A subfield

Unit 3: Principal Ideals and Quotient Rings

Lecture Input: "Composition tables for finite quotient rings" is your syllabus objective. In IIT JAM, this translates to constructing $\mathbb{Z}_2[x]/(x^2 + x + 1)$.

Q5: Structure of Quotient Ring (IIT JAM Standard)

Question: Let $R = \frac{\mathbb{Z}_2[x]}{(x^2+x+1)}$. How many elements does R have?

(A) 2 (B) 3 (C) 4 (D) 8

Solution:

- Step 1: $\mathbb{Z}_2$ is the field $\{0,1\}$. A polynomial ring modulo a degree-2 polynomial.
- Step 2: The remainder upon division by $x^2 + x + 1$ can be of the form $a + bx$, where $a, b \in \mathbb{Z}_2$.
- Step 3: Possible pairs: $0, 1, x, 1 + x$.
- Answer: (C) 4
    - *Extension:* Check if it's a field? $x^2 + x + 1$ has no root in $\mathbb{Z}_2$ (0+0+1=1, 1+1+1=1). Since it's irreducible, the quotient is a field of order 4 (Galois Field GF(4)).

Q6: Is $\mathbb{Z}[x]/(x)$ an integral domain?

Solution:

- (x) is the ideal of polynomials with zero constant term.
- The quotient $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$ (by the evaluation homomorphism $f(x) \to f(0)$).
- $\mathbb{Z}$ is an integral domain.
- Answer: Yes.

Unit 4: Homomorphism of Rings

Lecture Input: The "Fundamental Theorems" are best understood via the *Kernel*.

Q7: First Isomorphism Theorem Application

Question: Define $\phi: \mathbb{Z}[x] \to \mathbb{Z}$ by $\phi(f(x)) = f(0)$. What is $\mathbb{Z}[x]/\ker(\phi)$ isomorphic to?

(A) $\mathbb{Z}[x]$ (B) $\mathbb{Z}$ (C) $\mathbb{Q}$ (D) $\mathbb{Z}_2$

Solution:

- Step 1: ϕ is the evaluation at 0. It is a homomorphism.
- Step 2: $\ker(\phi)$ = set of polynomials with zero constant term = (x).
- Step 3: By the Fundamental Theorem, $\mathbb{Z}[x]/(x) \cong \text{Im}(\phi)$.

- Step 4: The image is all integers (since constant polynomials map to integers).
- Answer: (B) $\mathbb{Z}$

Q8: Maximal vs Prime Ideals

Question: In $\mathbb{Z}$, the ideal $6\mathbb{Z}$ is:

(A) Maximal but not Prime

(B) Prime but not Maximal

(C) Neither Prime nor Maximal

(D) Both Prime and Maximal

Solution:

- Step 1: $6\mathbb{Z}$ is not maximal because $6\mathbb{Z} \subset 2\mathbb{Z} \subset \mathbb{Z}$.
- Step 2: Check Prime: $ab \in 6\mathbb{Z}$ means 6 divides ab. Does 6 divide a or 6 divide b? No: e.g., $2 \times 3 = 6 \in 6\mathbb{Z}$, but $2 \notin 6\mathbb{Z}$ and $3 \notin 6\mathbb{Z}$.
- Step 3: Therefore, it fails the prime test.
- Answer: (C) Neither Prime nor Maximal

Unit 5: Rings of Polynomials

Lecture Input: The Division Algorithm is the engine of Euclidean domains.

Q9: Division Algorithm in $F[x]$

Question: Divide $f(x) = x^3 + 2x^2 + 1$ by $g(x) = x^2 + 1$ in $\mathbb{Z}_3[x]$ (integers mod 3). Find the remainder.

(A) $x + 2$ (B) $2x + 1$ (C) $2x + 2$ (D) $x + 1$

Solution:

- Step 1: In $\mathbb{Z}_3$, coefficients are 0,1,2 (where 2 = -1).
- Step 2: Long division:
  - Leading term: $x^3/x^2 = x$. Multiply: $x(x^2 + 1) = x^3 + x$. Subtract: $(x^3 + 2x^2 + 1) - (x^3 + x) = 2x^2 - x + 1$. Since -1 = 2, this is $2x^2 + 2x + 1$.
  - Next term: $2x^2/x^2 = 2$. Multiply: $2(x^2 + 1) = 2x^2 + 2$.
    Subtract: $(2x^2 + 2x + 1) - (2x^2 + 2) = 2x - 1$.
- Step 3: Since -1 = 2, remainder = $2x + 2$.
- Answer: (C) $2x + 2$

Q10: Reducibility over Fields

Question: Which polynomial is irreducible over the field $\mathbb{Q}$?

(A) $x^2 - 2$ (B) $x^2 + 4x + 4$ (C) $x^3 - x$ (D) $x^4 - 16$

Solution:

- Step 1: (A) Roots are $\pm\sqrt{2}$, not rational. Degree 2 with no rational root implies irreducible over $\mathbb{Q}$ (by Rational Root Theorem).
- Step 2: (B) $(x+2)^2$ is reducible.
- Step 3: (C) $x(x-1)(x+1)$ is reducible.
- Step 4: (D) $(x^2-4)(x^2+4)$ is reducible.
- Answer: (A)

DIFFERENTIAL EQUATIONS

Q1: Exact Differential Equation

Question:

Check whether the following differential equation is exact. If yes, solve it:

$$(2xy + y^2)dx + (x^2 + 2xy)dy = 0$$

Answer:

- Here $M = 2xy + y^2$, $N = x^2 + 2xy$
- $\frac{\partial M}{\partial y} = 2x + 2y, \frac{\partial N}{\partial x} = 2x + 2y \rightarrow$ Equal $\rightarrow$ Exact.
- Solution: $\int M dx = x^2y + xy^2 + f(y)$
- Compare with $N \rightarrow f'(y) = 0 \Rightarrow f(y) = c$
- Final: $x^2y + xy^2 = C$

Q2: Bernoulli's Equation

Question:

Solve:

$$\frac{dy}{dx} + y = y^2 \sin\, x$$

Answer:

- Divide by y^2: $y^{-2}\frac{dy}{dx} + y^{-1} = \sin\, x$
- Let $v = y^{-1} \Rightarrow \frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$
- Substituting: $-\frac{dv}{dx} + v = \sin\, x \Rightarrow \frac{dv}{dx} - v = -\sin\, x$

- Integrating factor e^{-x} → solution: $v = Ce^x + \frac{\sin x + \cos x}{2}$
- Back substitute $v = 1/y$

Q3: IIT JAM Style – Wronskian

Question:

If $y_1 = e^x$ and $y_2 = e^{2x}$ are solutions of a linear ODE, find their Wronskian.

Answer:

$$W = y_1 y_2' - y_2 y_1' = e^x \cdot 2e^{2x} - e^{2x} \cdot e^x = 2e^{3x} - e^{3x} = e^{3x} \neq 0$$

Hence they are linearly independent.

ADDITIONAL QUESTIONS: DIFFERENTIAL EQUATIONS

Unit 1: First Order First Degree

Q4: Linear Differential Equation

Question:

Solve: $\frac{dy}{dx} + y\cot x = \csc x$

Solution:

- Integrating factor (I.F.) = $e^{\int \cot x \, dx} = e^{\ln|\sin x|} = \sin x$
- Solution: $y \cdot \sin x = \int \csc x \cdot \sin x \, dx = \int 1 \, dx = x + C$
- Answer: $y\sin x = x + C$

Q5: Equation Reducible to Exact (Using $\frac{1}{Mx+Ny}$)

Question:

Solve: $(x^2 + y^2)dx + (x^2 - xy)dy = 0$

Solution:

- $M = x^2 + y^2$, $N = x^2 - xy$
- $\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 2x - y$ → Not exact.
- Compute $Mx + Ny = x(x^2 + y^2) + y(x^2 - xy) = x^3 + xy^2 + x^2y - xy^2 = x^3 + x^2y = x^2(x + y)$
- I.F. = $\frac{1}{Mx+Ny} = \frac{1}{x^2(x+y)}$

- Multiply and solve (exact now) → Final implicit solution:

  Answer: $\frac{x^2}{2}+\frac{y^2}{2}-\frac{y^3}{3x}=C$

Q6: IIT JAM – Orthogonal Trajectories

Question:

Find the orthogonal trajectories of the family of curves $y^2 = 4ax$.

Solution:

- Differentiate: $2y\frac{dy}{dx}=4a \Rightarrow \frac{dy}{dx}=\frac{2a}{y}$
- Eliminate a: From original, $a=\frac{y^2}{4x}$, so $\frac{dy}{dx}=\frac{2(y^2/4x)}{y}=\frac{y}{2x}$
- For orthogonal trajectories, replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$:

$$-\frac{dx}{dy}=\frac{y}{2x} \Rightarrow 2xdx+ydy=0$$

- Integrate: $x^2+\frac{y^2}{2}=C$
- Answer: $2x^2+y^2=C'$

Unit 2: First Order But Not First Degree

Q7: Equation Solvable for p

Question:

Solve: $y=2px+p^2$ where $p=\frac{dy}{dx}$

Solution:

- Differentiate w.r.t x: $\frac{dy}{dx}=2p+2x\frac{dp}{dx}+2p\frac{dp}{dx}$
- But $\frac{dy}{dx}=p$, so:

$$p=2p+(2x+2p)\frac{dp}{dx}$$

$$\Rightarrow -p=2(x+p)\frac{dp}{dx}$$

$$\Rightarrow \frac{dx}{dp}+\frac{2}{p}x=-2$$

- Linear in x: I.F. = $e^{\int \frac{2}{p}dp}=p^2$
- Solution: $xp^2=-\frac{2}{3}p^3+C$

- Substitute back into original to eliminate p (general solution and singular solution).

  Answer: General: $y = 2x\left(\frac{C}{x}\right)^{1/2} + \frac{C}{x}$, Singular: $y = -x^2$

Q8: Clairaut's Equation

Question:

Solve: $y = xy' + (y')^2$

Solution:

- Clairaut form: $y = xp + p^2$
- General solution: Replace p with C: $y = Cx + C^2$
- Singular solution: Eliminate p from $y = xp + p^2$ and $0 = x + 2p \to p = -x/2$
- Substitute back: $y = x(-x/2) + (x^2/4) = -x^2/4$
- Answer: General: $y = Cx + C^2$; Singular: $4y + x^2 = 0$

Unit 3: Higher Order Linear (Constant Coefficients)

Q9: Homogeneous Equation

Question:

Solve: $y''' - 3y'' + 3y' - y = 0$

Solution:

- Auxiliary equation: $m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0$
- Roots: $m = 1,1,1$ (triple root)
- Answer: $y = (C_1 + C_2x + C_3x^2)e^x$

Q10: Non-Homogeneous – Method of Undetermined Coefficients

Question:

Solve: $y'' + 3y' + 2y = e^x$

Solution:

- Auxiliary: $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$
- C.F. = $C_1e^{-x} + C_2e^{-2x}$
- P.I.: Assume $y_p = Ae^x$, substitute: $Ae^x + 3Ae^x + 2Ae^x = e^x \Rightarrow 6A = 1 \Rightarrow A = 1/6$
- Answer: $y = C_1e^{-x} + C_2e^{-2x} + \frac{1}{6}e^x$

Q11: IIT JAM – P.I. for $e^{ax}V$

Question:

Solve: $y'' - 2y' + y = xe^x$

Solution:

- Auxiliary: $m^2 - 2m + 1 = (m-1)^2 = 0 \to$ C.F. = $(C_1 + C_2x)e^x$
- P.I. = $\frac{1}{(D-1)^2}xe^x = e^x\frac{1}{D^2}x$ (using shifting property)
- $\frac{1}{D^2}x = \int\int x\,dxdx = \frac{x^3}{6}$
- P.I. = $e^x \cdot \frac{x^3}{6}$
- Answer: $y = (C_1 + C_2x)e^x + \frac{x^3}{6}e^x$

Unit 4: Cauchy-Euler and Variation of Parameters

Q12: Cauchy-Euler Equation

Question:

Solve: $x^2y'' - 3xy' + 3y = 0$

Solution:

- Let $y = x^m$, then $m(m-1) - 3m + 3 = 0 \Rightarrow m^2 - 4m + 3 = 0 \Rightarrow m = 1,3$
- Answer: $y = C_1x + C_2x^3$

Q13: Variation of Parameters

Question:

Solve: $y'' + y = \sec\ x$

Solution:

- C.F.: $y_c = C_1\cos\ x + C_2\sin\ x$
- Wronskian: $W = \cos\ x \cdot \cos\ x - \sin\ x \cdot (-\sin\ x) = 1$
- $u_1' = -\frac{y_2f}{W} = -\frac{\sin\ x\sec\ x}{1} = -\tan\ x \to u_1 = \ln\ |\cos\ x|$
- $u_2' = \frac{y_1f}{W} = \frac{\cos\ x\sec\ x}{1} = 1 \to u_2 = x$
- Answer: $y = C_1\cos\ x + C_2\sin\ x + \cos\ x\ln\ |\cos\ x| + x\sin\ x$

Unit 5: System of ODEs (IIT JAM Level)

Q14: Simultaneous Differential Equations

Question:

Solve:

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = x + 2y$$

Solution:

- Matrix form: $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
- Eigenvalues: $\lambda^2 - 4\lambda + 1 = 0 \Rightarrow \lambda = 2 \pm \sqrt{3}$
- Eigenvectors: For $\lambda = 2 + \sqrt{3}$: $(\sqrt{3}, 1)$; For $\lambda = 2 - \sqrt{3}$: $(-\sqrt{3}, 1)$
- Answer:

$$x(t) = C_1\sqrt{3}e^{(2+\sqrt{3})t} - C_2\sqrt{3}e^{(2-\sqrt{3})t}$$

$$y(t) = C_1 e^{(2+\sqrt{3})t} + C_2 e^{(2-\sqrt{3})t}$$

Q15: IIT JAM – Application (Mixing Problem)

Question:

A tank initially contains 50 L of water with 10 kg of salt. Brine with 2 kg/L salt enters at 5 L/min. The mixture leaves at 5 L/min. Find the amount of salt at time t.

Solution:

- Let $S(t)$ = salt in tank (kg). Volume constant = 50 L.
- Rate in = $5 \times 2 = 10$ kg/min
- Rate out = $\frac{S}{50} \times 5 = \frac{S}{10}$ kg/min
- Differential equation: $\frac{dS}{dt} = 10 - \frac{S}{10}$
- Solve: $\frac{dS}{dt} + \frac{1}{10}S = 10$
- I.F. = $e^{t/10} \rightarrow Se^{t/10} = \int 10e^{t/10}dt = 100e^{t/10} + C$
- $S(t) = 100 + Ce^{-t/10}$
- Initial: $S(0) = 10 \Rightarrow 10 = 100 + C \Rightarrow C = -90$
- Answer: $S(t) = 100 - 90e^{-t/10}$ kg

GROUP THEORY

Q1: Subgroup Criterion

Question:

Check whether $H = \{0,2,4\}$ is a subgroup of $\mathbb{Z}_6$ under addition modulo 6.

Answer:

- Closure: $2 + 4 = 0, 2 + 2 = 4, 4 + 4 = 2 \rightarrow$ closed
- Identity: $0 \in H$
- Inverse: 2's inverse is 4, 4's inverse is 2, 0's inverse is 0
- Yes, it is a subgroup of order 3.

Q2: Normal Subgroup

Question:

Show that every subgroup of index 2 is normal.

Answer:

Let H be a subgroup of G with $[G:H] = 2$.

Then left cosets are H and gH, right cosets are H and Hg.

Since gH and Hg both partition G and contain g, they must be equal.

Thus $gH = Hg$ for all $g \notin H$. For $g \in H$, trivially holds. Hence H is normal.

Q3: IIT JAM – Lagrange's Theorem Application

Question:

If G is a group of order 15, what are the possible orders of subgroups?

Answer:

By Lagrange's theorem, order of any subgroup divides 15.

Possible orders: 1, 3, 5, 15.

GROUP THEORY – IIT JAM LEVEL PRACTICE SET

Question 1

If G is a group of prime order p, then G has:

(A) no proper non-trivial subgroups

(B) exactly one proper non-trivial subgroup

(C) exactly two proper non-trivial subgroups

(D) infinitely many subgroups

Solution: By Lagrange's theorem, order of any subgroup divides $|G| = p$. Since p is prime, the only divisors are 1 and p. Hence the only subgroups are {e} and G itself. No proper non-trivial subgroups exist. Answer: (A)

Question 2

Number of proper subgroups of Z_6 (integers modulo 6 under addition) is:

(A) 2

(B) 3

(C) 4

(D) 5

Solution: $Z_6 = \{0,1,2,3,4,5\}$. Subgroups correspond to divisors of 6. Subgroups: {0} (order 1), {0,3} (order 2), {0,2,4} (order 3), Z_6 (order 6). Proper subgroups are those not equal to G itself: {0}, {0,3}, {0,2,4} → 3 proper subgroups. Answer: (B)

Question 3

Consider the group of 2×2 non-singular matrices under matrix multiplication over Z_5. Find the inverse of [1, 2; 3, 4].

Solution: For a 2×2 matrix A = [[a, b], [c, d]], inverse = $(ad-bc)^{-1}$ [[d, -b], [-c, a]] over the field.

$ad - bc = 1\times4 - 2\times3 = 4 - 6 = -2 \equiv 3 \pmod 5$.

Inverse of 3 mod 5 is 2 (since $3\times2=6\equiv1$).

So A^{-1} = 2 × [[4, -2], [-3, 1]] = 2 × [[4, 3], [2, 1]] mod 5 = [[8, 6], [4, 2]] ≡ [[3, 1], [4, 2]] mod 5. Answer: [[3,1],[4,2]]

Question 4

Let G be a group of order 289. Then G is:

(A) cyclic but not abelian

(B) abelian but not cyclic

(C) cyclic and hence abelian

(D) not necessarily abelian

Solution: $289 = 17^2$. Any group of order p^2 (p prime) is abelian. Moreover, it is either cyclic Z_{p^2} or $Z_p \times Z_p$. Answer: (C)

Question 5

Which of the following groups is cyclic?

(A) $Z_2 \times Z_3$

(B) $Z_2 \times Z_2$

(C) $Z_3 \times Z_3$

(D) $Z_2 \times Z_4$

Solution: $Z_m \times Z_n$ is cyclic if and only if gcd(m,n)=1.

(A) gcd(2,3)=1 → $Z_2 \times Z_3 \cong Z_6$ cyclic.

(B) gcd(2,2)=2 → not cyclic.

(C) gcd(3,3)=3 → not cyclic.

(D) gcd(2,4)=2 → not cyclic. Answer: (A)

Question 6

If G is a finite group and Z_{10} is a homomorphic image of G, then:

(A) 10 divides |G|

(B) |G| divides 10

(C) G is cyclic

(D) G is abelian

Solution: If there exists a homomorphism from G onto Z_{10}, then by First Isomorphism Theorem, $Z_{10} \cong G/\ker \varphi$. Hence $|G| = |\ker \varphi| \times 10$, so 10 divides |G|. Answer: (A)

Question 7

The number of all subgroups of the group (Z_{60}, +) is:

(A) 6

(B) 8

(C) 10

(D) 12

Solution: For cyclic group Z_n, number of subgroups = number of divisors of n. $60 = 2^2 \times 3 \times 5$. Number of divisors = (2+1)(1+1)(1+1) = 3×2×2 = 12. Answer: (D)

Question 8

Which of the following statements is correct?

(A) Every subgroup of an abelian group is normal

(B) Every subgroup of a non-abelian group is normal

(C) Center of a group is not necessarily a subgroup

(D) Intersection of two subgroups is not necessarily a subgroup

Solution: (A) is true: In an abelian group, $gHg^{-1} = H$ for all g.

(B) False: Non-abelian groups have non-normal subgroups (e.g., S_3).

(C) False: Center $Z(G)$ is always a subgroup.

(D) False: Intersection of subgroups is always a subgroup. Answer: (A)

Question 9

Let G be a group of order 6, and H be a subgroup of G such that $1 < |H| < 6$. Which is correct?

(A) H is always cyclic

(B) H is always normal

(C) H may not be normal

(D) G is always abelian

Solution: For $|G|=6$, possible proper subgroups have orders 2 or 3. Any subgroup of prime order is cyclic. In S_3 (order 6), subgroups of order 3 are normal, but subgroups of order 2 are not normal. So H may not be normal. Answer: (C)

Question 10

If $(G, \cdot)$ is a group such that $(ab)^{-1} = a^{-1}b^{-1}$ for all $a,b \in G$, then G is:

(A) non-abelian

(B) abelian

(C) cyclic

(D) finite

Solution: $(ab)^{-1} = b^{-1}a^{-1}$. Given $b^{-1}a^{-1} = a^{-1}b^{-1}$ for all a,b. Taking inverses: $ab = ba$. Hence G is abelian. Answer: (B)

Question 11 (IIT JAM 2023 Style)

Let $A \subset Z$ with $0 \in A$. For $r,s \in Z$, define $rA = \{ra : a \in A\}$, $rA + sA = \{ra + sb : a,b \in A\}$. Which condition implies that A is a subgroup of $(Z,+)$?

(A) $-2A \subset A$, $A + A = A$

(B) $A = -A$, $A + 2A = A$

(C) $A = -A$, $A + A = A$

(D) $2A \subset A$, $A + A = A$

Solution: For A to be a subgroup of Z (additive):

- $0 \in A$ (given)
- $A + A = A$ gives closure under addition
- $A = -A$ gives closure under inverses

  Thus condition (C) is both necessary and sufficient. Answer: (C)

Question 12

Let G be a cyclic group of order 12. The number of generators of G is:

(A) 2

(B) 4

(C) 6

(D) 8

Solution: Number of generators of cyclic group of order n = $\varphi(n)$. $\varphi(12) = \varphi(2^2 \times 3) = 12 \times (1-1/2) \times (1-1/3) = 12 \times 1/2 \times 2/3 = 4$. Answer: (B)

Question 13

Let G be a group of order 30. Which of the following is NOT possible?

(A) G has a subgroup of order 15

(B) G has a subgroup of order 10

(C) G has a subgroup of order 6

(D) G has a subgroup of order 7

Solution: By Lagrange's theorem, order of any subgroup must divide $|G|=30$. 7 does not divide 30. Hence no subgroup of order 7 exists. Answer: (D)

Question 14

The center of a group G is defined as $Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}$. Which is true?

(A) Z(G) is always a normal subgroup of G

(B) Z(G) is always cyclic

(C) Z(G) is always abelian but not necessarily normal

(D) Z(G) is always trivial

Solution: Z(G) is always a subgroup. For any $z \in Z(G)$ and any $g \in G$, $gzg^{-1} = z \in Z(G)$. So $Z(G) \triangleleft G$. Answer: (A)

Question 15

A subgroup H of a group G is normal if and only if:

(A) $gHg^{-1} \subseteq H$ for all $g \in G$

(B) $gHg^{-1} = H$ for all $g \in G$

(C) both (A) and (B)

(D) H is abelian

Solution: By definition, $H \triangleleft G$ if $gHg^{-1} = H$ for all $g \in G$. However, if G is finite, $gHg^{-1} \subseteq H$ implies equality. The complete correct condition is $gHg^{-1} = H$. Answer: (C)

Question 16

Which of the following is a simple group?

(A) Z_4

(B) S_3

(C) Z_5

(D) D_4

Solution: A simple group has no proper non-trivial normal subgroups. Z_5 is cyclic of prime order, hence has no proper subgroups at all, so it is simple. Z_4 has subgroup of order 2 (normal), S_3 has normal subgroup A_3, D_4 has normal subgroups. Answer: (C)

Question 17

Let $\varphi: G \to G'$ be a group homomorphism. Then ker φ is:

(A) a subgroup of G but not necessarily normal

(B) a normal subgroup of G

(C) a subgroup of G'

(D) always trivial

Solution: Kernel of a homomorphism is always a normal subgroup of the domain G. Answer: (B)

Question 18

A group of order 15 is:

(A) cyclic

(B) non-abelian

(C) simple

(D) not cyclic

Solution: $15 = 3 \times 5$. By a theorem, any group of order pq with primes $p<q$ and p not dividing q-1 is cyclic. Here 3 does not divide 4, so any group of order 15 is cyclic (hence abelian). Answer: (A)

Question 19

Let $G = S_3$ (symmetric group on 3 letters). The number of elements satisfying $x^2 = e$ is:

(A) 2

(B) 3

(C) 4

(D) 6

Solution: In S_3, elements: identity e ($e^2=e$), transpositions (12),(13),(23) each square to identity, and 3-cycles (123),(132) have order 3. So 1 + 3 = 4 elements satisfy $x^2 = e$. Answer: (C)

Question 20

If G is a finite group and H is a subgroup of index 2, then:

(A) H is always normal in G

(B) H may not be normal in G

(C) H is always cyclic

(D) H is always abelian

Solution: Any subgroup of index 2 is always normal in G. Because left cosets and right cosets coincide. Answer: (A)

REAL ANALYSIS

Q1: Convergence of Sequence

Question:

Test the convergence of $a_n = \frac{n^2+1}{2n^2+3}$

Answer:

$$\lim_{n\to\infty} \frac{n^2+1}{2n^2+3} = \frac{1}{2}$$

Hence the sequence converges to $1/2$.

Q2: Cauchy Criterion for Series

Question:

Check convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using Cauchy's criterion.

Answer:

For $m > n$,

$$| S_m - S_n |= \frac{1}{(n+1)^2} + \cdots + \frac{1}{m^2} < \int_n^{\infty} \frac{1}{x^2} dx = \frac{1}{n}$$

Given $\epsilon > 0$, choose $N > 1/\epsilon$. Then for $m, n > N$, $| S_m - S_n |< \epsilon$.

Hence convergent (p-series with p=2).

Q3: Uniform Continuity

Question:

Is $f(x) = 1/x$ uniformly continuous on $(0,1)$?

Answer:

No. Let $\epsilon = 1$, for any $\delta > 0$, choose $x = \delta/2$, $y = \delta/4$, then $| x - y |= \delta/4 < \delta$ but $| f(x) - f(y) |= 2/\delta > 1$ for small δ. So not uniformly continuous.

TOPIC 1: SEQUENCES OF REAL NUMBERS

Question 1

Let $\{a_n\}$ be a sequence of real numbers defined by

$$a_1 = 1, a_{n+1} = \frac{2a_n + 3}{4}, n \geq 1$$

Then $\lim_{n\to\infty} a_n$ is equal to:

(A) 0

(B) 1

(C) 3/2

(D) Does not exist

Solution:

Let $\lim_{n\to\infty} a_n = L$. Then taking limit on both sides:

$$L = \frac{2L + 3}{4}$$

$$4L = 2L + 3 \implies 2L = 3 \implies L = \frac{3}{2}$$

Answer: (C) 3/2

Question 2

Which of the following sequences is not convergent?

(A) $a_n = \frac{n^2+3n+1}{2n^2-n+5}$

(B) $a_n = \frac{\sin n}{n}$

(C) $a_n = (-1)^n \left(1-\frac{1}{n}\right)$

(D) $a_n = \frac{2^n}{n!}$

Solution:

(A) $\lim_{n\to\infty} \frac{n^2}{2n^2} = \frac{1}{2} \rightarrow$ convergent

(B) $|\frac{\sin n}{n}| \leq \frac{1}{n} \rightarrow 0 \rightarrow$ convergent (Squeeze theorem)

(C) $a_{2n} \rightarrow 1, a_{2n+1} \rightarrow -1 \rightarrow$ two different subsequential limits $\rightarrow$ divergent

(D) $\frac{2^n}{n!} \rightarrow 0 \rightarrow$ convergent

Answer: (C)

Question 3

Let $a_n = \sqrt{n+1} - \sqrt{n}$. Then $\lim_{n\to\infty} a_n$ is:

(A) 0

(B) 1

(C) ∞

(D) Does not exist

Solution:

Rationalize:

$$a_n = \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \rightarrow 0$$

Answer: (A) 0

Question 4 (IIT JAM Previous Year Pattern)

Let $\{a_n\}$ be a sequence such that $a_{n+1} = \sqrt{2+a_n}$ with $a_1 = \sqrt{2}$. Then the sequence is:

(A) decreasing and bounded below

(B) increasing and bounded above

(C) decreasing and bounded above

(D) increasing and bounded below

Solution:

Check: $a_1 = \sqrt{2} \approx 1.414$, $a_2 = \sqrt{2+\sqrt{2}} \approx 1.848$, $a_3 \approx 1.962$

So sequence is increasing.

Check boundedness: $a_n < 2$ (can be proved by induction). So bounded above by 2.

Answer: (B) increasing and bounded above

Question 5

If $a_n = \left(1+\frac{1}{n}\right)^n$, then $\lim_{n\to\infty} a_n$ is:

(A) 1

(B) e

(C) ∞

(D) 0

Solution:

This is the standard definition of e.

Answer: (B) e

TOPIC 2: SERIES OF REAL NUMBERS

Question 6

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if:

(A) $p > 1$

(B) $p \geq 1$

(C) $p < 1$

(D) $p \leq 1$

Solution:

This is the p-series test. Converges for $p > 1$, diverges for $p \leq 1$.

Answer: (A) $p > 1$

Question 7

Test the convergence of $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Solution:

Use Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n+1}{(n+1)^{n+1}} \cdot n^n$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

Hence the series converges.

Answer: Convergent

Question 8

Which of the following series converges?

(A) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(B) $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

(C) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

(D) $\sum_{n=1}^{\infty} \frac{1}{n^{0.5}}$

Solution:

Options (A), (B), (D) are the same: $p = 1/2 < 1 \rightarrow$ divergent

Option (C): $p = 3/2 > 1 \rightarrow$ convergent

Answer: (C)

Question 9 (Alternating Series)

Test the convergence of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$.

Solution:

Alternating series test:

- $b_n = 1/n$ is decreasing
- $\lim_{n\to\infty} b_n = 0$

  Therefore, the series converges conditionally (this is the alternating harmonic series).

Answer: Conditionally convergent

Question 10

The radius of convergence of the power series $\sum_{n=0}^{\infty} n!\,x^n$ is:

(A) 0

(B) 1

(C) ∞

(D) e

Solution:

Using Ratio Test for radius:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty$$

So radius of convergence $R = 0$.

Answer: (A) 0

TOPIC 3: LIMIT AND CONTINUITY

Question 11

The function $f(x) = \frac{|x|}{x}$ at $x = 0$ is:

(A) Continuous

(B) Discontinuous with removable discontinuity

(C) Discontinuous with jump discontinuity

(D) Discontinuous with infinite discontinuity

Solution:

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Left-hand limit = -1, Right-hand limit = 1. Jump discontinuity at $x = 0$.

Answer: (C)

Question 12

Let $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Then f is:

(A) Discontinuous at $x = 0$

(B) Continuous at $x = 0$

(C) Differentiable at $x = 0$

(D) Neither continuous nor differentiable

Solution:

$\lim_{x\to 0} \frac{\sin x}{x} = 1 = f(0)$. Hence continuous at $x = 0$.

Also, $f'(0)$ exists (equals 0).

Answer: (B) and (C)

Question 13

If $f(x) = x\sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$, then at $x = 0$:

(A) f is continuous and differentiable

(B) f is continuous but not differentiable

(C) f is not continuous

(D) f is differentiable but derivative is not continuous

Solution:

$\lim_{x\to 0} x\sin(1/x) = 0$ (bounded × 0). So continuous.

Derivative:

$$f'(0) = \lim_{h \to 0} \frac{h\sin(1/h) - 0}{h} = \lim_{h \to 0} \sin(1/h)$$

This limit does not exist → not differentiable.

Answer: (B)

TOPIC 4: DIFFERENTIABILITY

Question 14

Rolle's theorem is applicable to $f(x) = x^2 - 4x + 3$ on the interval:

(A) [0, 4]

(B) [1, 3]

(C) [0, 2]

(D) [2, 4]

Solution:

Rolle's theorem requires $f(a) = f(b)$.

Check: $f(1) = 1 - 4 + 3 = 0$, $f(3) = 9 - 12 + 3 = 0$. So on $[1, 3]$, $f(1) = f(3) = 0$.

Answer: (B)

Question 15

Let $f(x) = x^3 - 3x + 1$. The number of distinct real roots of $f(x) = 0$ is:

(A) 0

(B) 1

(C) 2

(D) 3

Solution:

$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$. Critical points at $x = -1$ and $x = 1$.

$$f(-1) = -1 + 3 + 1 = 3 > 0$$

$$f(1) = 1 - 3 + 1 = -1 < 0$$

Since $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$, and sign changes occur, there are 3 real roots.

Answer: (D) 3

Question 16

Mean Value Theorem guarantees a point $c \in (0,2)$ such that $f'(c) = \frac{f(2)-f(0)}{2}$ for:

(A) $f(x) =| x - 1 |$

(B) $f(x) = \frac{1}{x-1}$

(C) $f(x) = x^{2/3}$

(D) $f(x) = x^2$

Solution:

MVT requires f to be continuous on [0,2] and differentiable on (0,2).

(A) $f(x) =| x - 1 |$ is not differentiable at $x = 1$ → fails

(B) $f(x) = 1/(x - 1)$ is not continuous at $x = 1$ → fails

(C) $f(x) = x^{2/3}$ is not differentiable at $x = 0$ → fails

(D) $f(x) = x^2$ is continuous and differentiable on the interval → satisfies

Answer: (D)

TOPIC 5: SEQUENCE AND SERIES OF FUNCTIONS

Question 17

The sequence of functions $f_n(x) = x^n$ on $[0, 1]$ converges:

(A) Pointwise to 0 for all $x \in [0,1]$

(B) Pointwise to 0 for $x \in [0,1)$ and to 1 at $x = 1$

(C) Uniformly to 0

(D) Uniformly to the zero function

Solution:

For $x \in [0,1)$: $x^n \to 0$

At $x = 1$: $1^n \to 1$

So pointwise limit: $f(x) = 0$ for $x \in [0,1)$, $f(1) = 1$

Not uniform because $\sup_{x\in[0,1)} | x^n - 0 |= 1$ does not approach 0.

Answer: (B)

Question 18

Which of the following is true about the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$?

(A) Converges uniformly on $(-1,1)$

(B) Converges uniformly on $[-1/2,1/2]$

(C) Converges uniformly on $(-1,1]$

(D) Converges uniformly on $[-1,1)$

Solution:

The series has radius of convergence $R = 1$.

On $[-1/2,1/2]$, $|x| \leq 1/2$, so by Weierstrass M-test with $M_n = (1/2)^n/n$, the series converges uniformly.

Answer: (B)

TOPIC 6: RIEMANN INTEGRATION

Question 19

Which of the following functions is Riemann integrable on $[0,1]$?

(A) $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

(B) $f(x) = \begin{cases} 1, & x = \frac{1}{n}, n \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$

(C) $f(x) = \frac{1}{x}$

(D) $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution:

(A) Dirichlet function: discontinuous everywhere → not Riemann integrable

(B) Discontinuous only at countably many points → Riemann integrable (integral = 0)

(C) $1/x$ is unbounded on $[0,1]$ → not Riemann integrable

(D) Unbounded → not Riemann integrable

Answer: (B)

Question 20

$\int_0^1 x^2\,dx$ is equal to:

(A) 0

(B) 1/3

(C) 1/2

(D) 1

Solution:

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

Answer: (B) 1/3

TOPIC 7: FUNCTIONS OF SEVERAL VARIABLES

Question 21

For $f(x,y) = x^2 + y^2$, the directional derivative at $(1,1)$ in the direction of the vector $(1,0)$ is:

(A) 1

(B) 2

(C) $\sqrt{2}$

(D) 0

Solution:

$$\nabla f = (2x, 2y) \implies \nabla f(1,1) = (2,2)$$

Unit vector in direction $(1,0)$ is $\hat{u} = (1,0)$

$$D_u f = \nabla f \cdot \hat{u} = (2,2) \cdot (1,0) = 2$$

Answer: (B) 2

Question 22

The limit $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$:

(A) Exists and equals 0

(B) Exists and equals 1

(C) Does not exist

(D) Exists and equals ∞

Solution:

Along $y = x^2$:

$$\frac{x^2 \cdot x^2}{x^4 + x^4} = \frac{x^4}{2x^4} = \frac{1}{2}$$

Along $y = x$:

$$\frac{x^3}{x^4 + x^2} = \frac{x}{x^2 + 1} \to 0$$

Different paths give different limits → limit does not exist.

Answer: (C)

VECTOR CALCULUS

Q1: Gradient and Directional Derivative

Question:

Find the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ at $(1,1,1)$ in the direction of $\vec{v} = (1,2,2)$.

Answer:

- Gradient: $\nabla f = (2x, 2y, 2z) \Rightarrow \nabla f(1,1,1) = (2,2,2)$
- Unit vector $\hat{v} = \frac{(1,2,2)}{3}$
- Directional derivative = $\nabla f \cdot \hat{v} = \frac{2+4+4}{3} = \frac{10}{3}$

Q2: Verify Divergence Theorem

Question:

Verify divergence theorem for $\vec{F} = (x, y, z)$ over the unit cube $[0, 1]^3$.

Answer:

- Div $\vec{F} = 3$
- Volume integral: $\iiint_V 3\, dV = 3 \times 1 = 3$
- Surface integral: On each face, flux = 1 (e.g., $x = 1$ face: $\int_0^1 \int_0^1 1 \cdot dydz = 1$)
- 6 faces → total flux = 6 → Wait, that gives 6, not 3 — mistake? Let's check carefully:

  Actually for $x = 1$: $\vec{F} \cdot \hat{n} = 1$, area=1 → flux=1.

  For $x = 0$: $\vec{F} \cdot \hat{n} = 0$. Similarly for y and z.

  So total flux = 1 (x=1) + 1 (y=1) + 1 (z=1) = 3. Yes. Matches.

Q3: Stokes' Theorem (IIT JAM Level)

Question:

Use Stokes' theorem to evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (-y, x, z)$ and C is the circle $x^2 + y^2 = 1, z = 0$.

Answer:

- Curl $\vec{F} = (0,0,2)$
- Surface S: disk in xy-plane, normal $\hat{n} = (0,0,1)$
- $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \iint_S 2\, dS = 2 \times \text{Area} = 2\pi$
- Hence line integral = 2π.

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5. MATRICES (Semester I – Course 2, Unit 1 / Semester V – Course 12, Unit 4)

Q1: Cayley-Hamilton Theorem

Question:

For $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, verify Cayley-Hamilton theorem and find A^{-1}.

Answer:

- Characteristic equation: $\lambda^2 - 4\lambda + 3 = 0$
- $A^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$

- $A^2 - 4A + 3I = \begin{pmatrix} 5-8+3 & 4-4 \\ 4-4 & 5-8+3 \end{pmatrix} = 0 \rightarrow$ Verified.
- From theorem: $A^2 - 4A + 3I = 0 \Rightarrow A^{-1} = \frac{4I-A}{3} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$

Q2: Rank of Matrix

Question:

Find rank of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

Answer:

Rows 2 and 3 are multiples of row 1. Only one linearly independent row.

Rank = 1.

Q3: Diagonalization (IIT JAM)

Question:

Diagonalize $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Answer:

- Eigenvalues: $\lambda = 1,3$
- Eigenvectors: for $\lambda = 1$: $(-1,1)$; for $\lambda = 3$: $(1, 1)$
- $P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$
- Then $A = PDP^{-1}$